Asymptotic Behaviour of Spectrums for Elliptic Pseudo-differential Operators

by

Xing Wang

A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

Baltimore, Maryland

February, 2015

Abstract

The aim of this dissertation is to study the asymptotic behaviours of spectrums for Elliptic Pseudo-seudodifferential Operators. In this work we included two results. First we studied the Nodal set of Steklov Eigenfunctions and obtained a lower bound for its size. Then we refined an end pioint estimate of Laplace-Beltrami eigenfunctions restricted to totally geodesic submanifolds.

Primary Reader: Christopher D. Sogge

Secondary Reader: Hans Lindblad

Acknowledgments

I would first like to express my deepest gratitude to my advisor, Dr. Christopher D. Sogge, for years of consistent support, patient guidance and encouragement. This thesis could not have been completed without his help. His spirit and kindness have set up a good example for my life.

I would also like to express my gratitude to Dr. William P. Minicozzi and Dr. Hans Lindblad for being great teachers and for teaching me PDE and geometry. I would like to thank Jiuyi Zhu for many helpful discussions and fruitful collaboration. I would like to thank my friend Vitaly Lorman for going through my thesis and checking for possible mistakes. I would like to thank my colleagves Chengbo Wang, Xiaohua Yao, Xin Yu, Sinan Ariturk, Peng Shao, Xuehua Chen, Hongtan Sun, Min Xue, Jordan Paschke, Qiang Guang, Chuya Guo, Chenyang Su, Vitaly Lorman, Jonathan Beardsley, Stephen Cattell, Shanlin Huang, Po-Yao Chang, Yi Li and Yakun Xi, for their support of my work with insightful discussions and warm company. I would like to thank the faculty and staff in the math department at Johns Hopkins: Dr. Jian Kong, Dr. Richard Brown, Morris Hunt, Sabrina Raymond, Linda Buckner, Charlene Poole, Christina Bannon and many others. And most importantly, my family for their love. Above all, my parents and wife, Aihua Luo, Yulin Wang and Ying Zhang.

Contents

List of Figures

Chapter 1

Introduction

1.1 Topics and historic results

The asymptotic behaviours of spectrums are of vital importance to mathematics, thus these topics have been studied frequently. A lot of great achievements have been made by different mathematicians through years of work. Here I selected some general and deep results, conjectures that I'm especially interested in. I shall only comment on the problems of interest to me.

Suppose (M, g) is a closed Riemannian manifold, P is an elliptic pseudo-differential operator of order 1. In following cases, $P = \sqrt{-\triangle_g}$ or $P = \Lambda$ where Λ is the Dirichlet-to-Neumann operator (defined in the remark). There exists an orthonormal basis $\{\phi_j\}$ of eigenfunctions such that

$$\Lambda \phi_j = \lambda_j \phi_j, \quad \phi_j \in C^\infty(\mathcal{M}), \quad \int_{\mathcal{M}} \phi_j \phi_k \, dV_g = \delta_{jk}.$$

Here, the eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3, \cdots$, are ordered in ascending order counted with multiplicity.

Remark 1.1.1. Here $(\mathcal{N}, h)$ is a smooth Riemannian manifold with boundary $(\mathcal{M}, g)$, where $\dim\mathcal{N} = n+1$ and $h|_{\mathcal{M}} = g$. The Dirichlet-to-Neumann operator Λ is defined as

$$\Lambda f = \frac{\partial}{\partial \nu}(Hf)|_{\mathcal{M}}$$

for $f \in H^{\frac{1}{2}}(\mathcal{M})$. Hf is the harmonic extension of f, i.e.

$$\begin{cases} \triangle_h u(x) = 0, & x \in \mathcal{N}, \\ u(x) = f(x), & x \in \partial\mathcal{N} = \mathcal{M} \end{cases}$$

with $u = Hf$. Specially, the eigenfunctions of Λ are called Steklov eigenfunctions.

Topic 1. *Weyl's Law, which gives an asymptotic formula of the number $N(\lambda)$ of eigenvalues less than or equal to λ.*

We have the following asymptotic formula(see [CSF] or [LH4]):

$$N(\lambda) = C(P, M)\lambda^n + O(\lambda^{n-1}) \tag{1.1}$$

As a direct application to a flat torus with $P = \sqrt{-\triangle_{\mathbb{T}}^n}$, we get an asyptotic formula for the number of lattice points in large balls. Improving the lower order term is related to the L^∞-norm of eigenfunctions, see [SZ2].

Topic 2. *Growth of L^p norm of eigenfunctions.*

C. Sogge obtained the following estimates for $p \geq 2$, when (M, g) is a sphere with standard metric. Estimates are sharp for both operators, see [CSF], [CSHZ]:

$$\|\phi_\lambda\|_{L^p(\mathcal{M})} \lesssim (1+\lambda)^{\sigma(n,p)} \|\phi_\lambda\|_{L^2(\mathcal{M})}, \tag{1.2}$$

where

$$\sigma(n,p) = \begin{cases} n(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}, & \frac{2(n+1)}{n-1} \le p \le \infty, \\ \frac{n-1}{2}(\frac{1}{2}-\frac{1}{p}), & 2 \le p \le \frac{2(n+1)}{n-1}. \end{cases}$$

Topic 3. *Size of nodal sets, or in other words, the size of zeroes of eigenfunctions.*

1. *For $P = \sqrt{-\triangle_g}$, the famous Yau conjecture on nodal sets is (denoted as N_λ here) :*

$$c\lambda \le H^{n-1}(N_\lambda) \le C\lambda$$

   It was proved on analytic manifolds by Donelly, Fefferman, see [DF].

   For general manifolds, in two dimensions, the best bounds are $c\lambda \le H^{n-1}(N_\lambda) \le C\lambda^{\frac{3}{2}}$.(see [DF2], [JB], etc). In higher dimensions, the best bounds are $c\lambda^{\frac{3-n}{2}} \le H^{n-1}(N_\lambda) \le C\lambda^{C\sqrt{\lambda}}$. (see [CM],[SZ], [HS], [HS1], etc).

2. *For $P = \Lambda$, Recently, some remarkable progress has been made for the upper bound of the size of nodal sets for analytic manifolds. Bellova and Lin [BL] proved that if $\mathcal{N}$ is an analytic domain in $\mathbb{R}^{n+1}$, then the H^{n-1}-Hausdorff measure of nodal sets of Steklov eigenfunctions has an upper bound of $C\lambda^6$ with C depending only on $\mathcal{N}$. Later, Zelditch [Zel] improved their results and showed that the optimal upper bound for the nodal sets is $C\lambda$ for real analytic manifolds.*

Topic 4. *Growth of L^p norm for eigenfunctions restricted to submanifolds.*

1. *For $P = \sqrt{-\triangle_g}$. In [BGT], N. Burq, P. Gérard, and N. Tzvetkov obtained the following L^p estimates for eigenfunctions restricted to submanifolds: Let Σ be a smooth submanifold of dimension k. There exists a constant $C > 0$ such that for any φ_λ, we have*

$$\|\varphi_\lambda\|_{L^p(\Sigma)} \le C(1+\lambda)^{\rho(k,n)}\|\varphi_\lambda\|_{L^2(M)}$$

where

$$\rho(n-1,n)=\begin{cases}\frac{n-1}{2}-\frac{n-1}{p} & \textit{if } p_0=\frac{2n}{n-1}<p\leq+\infty\\ \frac{n-1}{4}-\frac{n-2}{2p} & \textit{if } 2\leq p<p_0=\frac{2n}{n-1}\end{cases}$$

$$\rho(n-2,n)=\tfrac{n-1}{2}-\tfrac{n-2}{p}\ \textit{if } 2<p\leq+\infty$$

$$\rho(k,n)=\tfrac{n-1}{2}-\tfrac{k}{p}\ \textit{if } 1\leq p\leq n-3$$

If $p=p_0=\frac{2n}{n-1}$ *and* $k=n-1$, *we have*

$$\|\varphi_\lambda\|_{L^p(\Sigma)}\leq C(1+\lambda)^{\frac{n-1}{2n}}log^{\frac{1}{2}}(\lambda)\|\varphi_\lambda\|_{L^2(M)}$$

and if p=2 and k=n-2, we have

$$\|\varphi_\lambda\|_{L^p(\Sigma)}\leq C(1+\lambda)^{\frac{1}{2}}log^{\frac{1}{2}}(\lambda)\|\varphi_\lambda\|_{L^2(M)}$$

These estimates are sharp in the general case except $(n,k,p)=(n,n-1,\frac{2n}{n-1})$ *and* $(n,k,p)=(n,n-2,2)$, *which have a log loss. Later on, Rui Hu gave another proof in [RH] of these estimates and showed the log loss for the case* $(n,k,p)=(n,n-1,\frac{2n}{n-1})$ *can be removed.*

2. *For* $P=\Lambda$, *so far as I know, there are no similar results about this topic yet.*

Of course, there are also other interesting topics about the spectrum of elliptic pseudo-differential operators, for example, the first eigenvalue problem, associated semiclassical (Wigner) measures, exploiting geometric information from eigenvalues (Can you hear the shape of a drum?), etc.

Improving the estimates above requires us to have a better and deeper understanding of the quantities and relations between them. Recently a lot of progress has been made, for example, to estimates in Topic 3 and $P=\sqrt{-\triangle_g}$:

In order to improve the above estimates we need to have a better understanding for the half wave operator at long time. Christopher Sogge, Steve Zelditch have a beautiful equivalent condition for the L^∞ norm when the manifold is analytic. They showed that we get $o(\lambda^{\frac{n-1}{2}})$ growth if and only if there's no self focal point. See [SZ2], [SZ3], [SZ4], [CS3], etc.

And for the flat torus and $P = \sqrt{-\triangle_{\mathbb{T}}^n}$, it is conjectured that one has uniform bounds:

$$\|\phi_\lambda\|_{L^p(\mathbb{T}^n)} \lesssim C_p \|\phi_\lambda\|_{L^2(\mathbb{T}^n)}, p < \frac{2n}{n-2} \tag{1.3}$$

and

$$\|\phi_\lambda\|_{L^p(\mathbb{T}^n)} \lesssim (1+\lambda)^{\delta(n,p)} \|\phi_\lambda\|_{L^2(\mathbb{T}^n)}, p > \frac{2n}{n-2} \tag{1.4}$$

J. Bourgain has made substantial progress on this, see [J.B].

1.2 Our results

In this work, we will prove the following two theorems.

Theorem 1.2.1. *Let ϕ_λ be a normalized Steklov eigenfunction and α be a regular value of ϕ_λ. Denote*

$$L_\lambda^\alpha = \{x \in \mathcal{M} | \phi_\lambda = \alpha\}.$$

Then there exists a positive constant $\epsilon(\mathcal{N})$ such that, for $|\alpha| < \epsilon(\mathcal{N})\lambda^{-\frac{n-1}{4}}$,

$$H^{n-1}(L_\lambda^\alpha) \geq C\lambda^{\frac{3-n}{2}}$$

with C depending only on $\mathcal{N}$.

An immediate consequence of Theorem 1 is the measure of nodal sets of Steklov eigenfunctions. Let

$$N_\lambda = \{x \in \mathcal{M} | \phi_\lambda = 0\}.$$

Corollary 1.2.2. *If* 0 *is a regular value of the Steklov eigenfunction* ϕ_λ*, then*

$$H^{n-1}(N_\lambda) \geq C\lambda^{\frac{3-n}{2}}$$

with C *depending only on* $\mathcal{N}$*.*

Remark 1.2.3. Quite different from the case for the Laplacian-Beltrami operator, the Dirichlet-to-Neumann operator is a non-local operator, which causes additional difficulties. Fortunately, since we are measuring the whole size of the nodal sets which can be considered as a "partial global" quantity, we were able to find a way to overcome the difficulty and carry the argument through.

Theorem 1.2.4. *Let* (M, g) *be a compact smooth Riemannian manifold of dimension* d*, and let* Σ *be a smooth totally geodesic submanifold of dimension* $d-2$*. There exists a constant* $C > 0$ *such that for any* φ_λ*, we have*

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \leq C(1+\lambda)^{\frac{1}{2}} \|\varphi_\lambda\|_{L^2(M)}$$

Remark 1.2.5. In our argument, we find a relation between the half wave operator on the manifold and the projection map P_μ to eigenspace with eigenvalue $\leq \mu$. With this relation, we are able to use the uniform boundedness of P_μ to get a bound without any log loss, unlike the earlier work of N. Burq, P. Gérard and N. Tzvetkov in [BGT]. We are now working on the general submanifold case.

Chapter 2

Nodal sets of Steklov eigenfunctions

2.1 Introduction

In this chapter, we consider the lower bound estimates for the Steklov eigenfunctions on a smooth Riemannian manifold $(\mathcal{N}, h)$ with boundary $(\mathcal{M}, g)$, where $\dim\mathcal{N} = n+1$ and $h|_{\mathcal{M}} = g$. The Steklov eigenvalue problem is formulated as

$$\begin{cases} \triangle_h \phi_\lambda(x) = 0, \quad x \in \mathcal{N}, \\ \frac{\partial \phi_\lambda}{\partial \nu}(x) = \lambda \phi_\lambda(x), \quad x \in \partial\mathcal{N} = \mathcal{M}. \end{cases}$$

Here, ν is a unit outer normal vector on $\mathcal{M}$. The Steklov eigenvalues can also be reduced to the boundary $\mathcal{M}$. Then the ϕ_λ becomes the eigenfunction of the Dirichlet-to-Neumann operator, i.e.

$$\Lambda \phi_\lambda = \lambda \phi_\lambda.$$

The Dirichlet-to-Neumann operator Λ is defined as

$$\Lambda f = \frac{\partial}{\partial \nu}(Hf)|_{\mathcal{M}}$$

for $f \in H^{\frac{1}{2}}(\mathcal{M})$. Hf is the harmonic extension of f, i.e.

$$\begin{cases} \triangle_h u(x) = 0, & x \in \mathcal{N}, \\ u(x) = f(x), & x \in \partial\mathcal{N} = \mathcal{M} \end{cases}$$

with $u = Hf$. Moreover, the operator Λ is a self-adjoint operator from $H^{\frac{1}{2}}(\mathcal{M})$ to $H^{-\frac{1}{2}}(\mathcal{M})$ and there exists an orthonormal basis $\{\phi_j\}$ of eigenfunctions such that

$$\Lambda\phi_j = \lambda_j\phi_j, \quad \phi_j \in C^\infty(\mathcal{M}), \quad \int_{\mathcal{M}} \phi_j\phi_k \, dV_g = \delta_{jk}.$$

The eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3, \cdots$, are ordered in ascending order with counted multiplicity. For simplicity, we choose $n+1$ as the dimension of $\mathcal{N}$, which is a little bit different from the previous work by [BL] and [Zel].

The nodal sets are zero level sets of eigenfunctions. We want to study the asymptotic behaviour of the size of nodal sets of Steklov eigenfunctions for large λ. Recently, some remarkable progress has been made for the upper bound of the size of nodal sets for analytic manifolds. Bellova and Lin [BL] proved that if $\mathcal{N}$ is an analytic domain in $\mathbb{R}^n$, then the H^{n-2}-Hausdorff measure of nodal sets of Steklov eigenfunctions has an upper bound of $C\lambda^6$ with C depending only on $\mathcal{N}$. Later on, Zelditch [Zel] improved their results and showed that the optimal upper bound for the nodal sets is $C\lambda$ for real analytic manifolds. The optimality can be seen from the case that the manifold is a ball.

So far, nothing seems to be known for the lower bound of the nodal sets of Steklov eigenfunctions, even for analytic manifolds. The main goal of our paper is to address the lower bound of nodal sets over general compact smooth manifolds.

Let's first briefly review the literature concerning the nodal sets of classical eigenfunctions. Let ϕ_λ be an L^2 normalized eigenfunctions of Laplacian-Beltrami on compact manifold $(\mathcal{M}, g)$ without boundary,

$$-\triangle_g \phi_\lambda = \lambda^2 \phi_\lambda$$

and let

$$N_\lambda = \{x \in \mathcal{M} | \phi_\lambda(x) = 0\}.$$

Yau conjectured that for any smooth manifold, one should control the upper and lower bound of nodal sets of classical eigenfunctions as

$$c\lambda \leq H^{n-1}(N_\lambda) \leq C\lambda$$

where C, c depend only on the manifold $\mathcal{M}$. The conjecture is only verified for real analytic manifold by Donnelly-Fefferman in [DF]. For smooth manifolds, the conjecture is still not settled. Much progress has been obtained towards the lower bound of nodal sets. Colding and Minicozzi [CM], Sogge and Zelditch [SZ] independently obtained that

$$H^{n-1}(N_\lambda) \geq C\lambda^{\frac{3-n}{2}}$$

for smooth manifolds. For other related works on lower bounds of nodal sets of classical eigenfunctions, see [M], [HL], [HS], [SZ1], etc, to just mention a few. The methods in [CM] and [SZ] are quite different. Specially, the method in [SZ] is based on a beautiful new integral formula about the L^1 norm of $|\nabla \phi_\lambda|$ on the nodal set and the L^1 norm of ϕ_λ on $\mathcal{M}$. Our goal is to adapt their idea to the setting of a non-local operator, i.e. Steklov eigenfunctions.

Denote the α-level sets of Steklov eigenfunctions by L^α_λ, that is,

$$L^\alpha_\lambda = \{x \in \mathcal{M} | \phi_\lambda = \alpha\}.$$

We are able to prove the following:

Theorem 2.1.1. *Let ϕ_λ be a normalized Steklov eigenfunction and α be a regular value of ϕ_λ. There exists a positive constant $\epsilon(\mathcal{N})$ such that, for $|\alpha| < \epsilon(\mathcal{N})\lambda^{-\frac{n-1}{4}}$,*

$$H^{n-1}(L_\lambda^\alpha) \geq C\lambda^{\frac{3-n}{2}}$$

with C depending only on $\mathcal{N}$.

An immediate consequence of Theorem 1 is the measure of nodal sets of Steklov eigenfunctions. Let

$$N_\lambda = \{x \in \mathcal{M} | \phi_\lambda = 0\}.$$

Corollary 2.1.2. *If 0 is a regular value of the Steklov eigenfunction ϕ_λ, then*

$$H^{n-1}(N_\lambda) \geq C\lambda^{\frac{3-n}{2}}$$

with C depending only on $\mathcal{N}$.

2.2 Preliminaries

In this section, we will review and prepare some general results needed in the proof of Theorem 1. First, we need the following result from [T].

Lemma 2.2.1. *The Dirichlet-to-Neumann operator Λ is an elliptic self-adjoint pseudo-differential operator of order 1 over $\mathcal{M}$. Moreover,*

$$\Lambda = \sqrt{-\triangle_g} \ mod \ OPS^0(\mathcal{M}).$$

Here, OPS^m denotes the pseudo-differential operator of order m. Since Λ is an elliptic self-adjoint pseudo-differential operator, by the general results in [SS] (see also the book of Sogge [**?**] or [**?**] for Laplacian-Beltrami operator), we have the following L^p norm estimates.

Lemma 2.2.2. *Let ϕ_λ be the Steklov eigenfunction. One has the sharp estimates, for $p \geq 2$,*

$$\|\phi_\lambda\|_{L^p(\mathcal{M})} \lesssim (1+\lambda)^{\sigma(n,p)} \|\phi_\lambda\|_{L^2(\mathcal{M})}, \tag{2.1}$$

where

$$\sigma(n,p) = \begin{cases} n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \leq p \leq \infty, \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases}$$

In the whole paper, the notation $A \lesssim B$ or $A \gtrsim B$ denotes $A \leq CB$ or $A \geq CB$ for some generic constant C which does not depend on λ. If we follow exactly the same argument as [SZ], which makes use of Lemma 2.2.2 for $p = \infty$, we can obtain L^p norm estimates for $p = 1$, that is,

$$\|\phi_\lambda\|_{L^1(\mathcal{M})} \gtrsim (1+\lambda)^{-\frac{n-1}{4}} \|\phi_\lambda\|_{L^2(\mathcal{M})}. \tag{2.2}$$

We also need the L^p bounds for the pseudo-differential operators.

Lemma 2.2.3. *Suppose $P \in OPS^m(\mathcal{M})$. Then*

$$\|P\phi_\lambda\|_{L^p(\mathcal{M})} \lesssim (1+\lambda)^m \|\phi_\lambda\|_{L^p(\mathcal{M})}, \quad \forall 1 < p < \infty.$$

Specifically,

$$\|\nabla_g^m \phi_\lambda\|_{L^p(\mathcal{M})} \lesssim (1+\lambda)^m \|\phi_\lambda\|_{L^p(\mathcal{M})}.$$

Proof. Define the operator $\tilde{P} := P(1+\Lambda)^{-m}$. Then $\tilde{P} \in OPS^0(\mathcal{M})$. By the boundedness of the zeroth pseudo-differential operator over $L^p(\mathcal{M})$ in [CSF] or [T], the lemma follows easily. $\square$

2.3 Lower bounds of nodal sets

We will obtain the lower bounds of α-level sets of Steklov eigenfunctions in this section. Since the Dirichlet-to-Neumann operator is a non-local operator, we do not need information from the manifold $(\mathcal{N}, h)$. In the following argument, all derivatives and calculations are performed with respect to the manifold $(\mathcal{M}, g)$. We first express the manifold $\mathcal{M}$ as the disjoint union

$$\mathcal{M} = \bigcup_{j=1}^{N_+(\lambda)} D^\alpha_{j,+} \cup \bigcup_{j=1}^{N_-(\lambda)} D^\alpha_{j,-} \cup L^\alpha_\lambda,$$

where $D^\alpha_{j,+}$ and $D^\alpha_{j,-}$ are the connected components of the sets $\{x \in \mathcal{M} | \phi_\lambda > \alpha\}$ and $\{x \in \mathcal{M} | \phi_\lambda < \alpha\}$. Using the same idea as in [SZ], we can treat each component separately and then add them up. For simplicity, we just deal with two components. The same argument carries out for many components. Denote

$$D^\alpha_+ = \{x \in \mathcal{M} | \phi_\lambda(x) > \alpha\}$$

and

$$D^\alpha_- = \{x \in \mathcal{M} | \phi_\lambda(x) < \alpha\}.$$

Without loss of generality, we may assume α to be nonnegative. Since α is assumed to be a regular value of ϕ_λ, then L^α_λ is a smooth submanifold in $\mathcal{M}$ and the boundary $\partial D^\alpha_\pm = L^\alpha_\lambda$. By the Green formula, for any $f \in C^\infty(\mathcal{M})$, we have

$$\int_{D^\alpha_\pm} div(f \nabla \phi_\lambda)\, dv_g = \int_{L^\alpha_\lambda} < f \nabla \phi_\lambda, \nu > \, ds \tag{2.3}$$

where ds is the surface measure on L_λ^α induced by the metric g on $\mathcal{M}$ and ν is the exterior unit normal vector on L_λ^α with respect to $D_\pm^\alpha$ respectively. Note the Green formula is taken on $\mathcal{M}$ with metric g. Since $\phi_\lambda \equiv \alpha$ on L_λ^α, then $< \nabla\phi_\lambda, \nu >= \pm|\nabla\phi_\lambda|$ on L_λ^α. Thus, (2.3) becomes

$$\int_{D_+^\alpha} div(f\nabla\phi_\lambda)\, dv_g = -\int_{L_\lambda^\alpha} f|\nabla\phi_\lambda|\, ds. \tag{2.4}$$

Similarly, we have

$$\int_{D_-^\alpha} div(f\nabla\phi_\lambda)\, dv_g = \int_{L_\lambda^\alpha} f|\nabla\phi_\lambda|\, ds. \tag{2.5}$$

By (2.4) and (2.5), we obtain

$$2\int_{L_\lambda^\alpha} f|\nabla\phi_\lambda|\, ds = \int_{D_-^\alpha} div(f\nabla\phi_\lambda)\, dv_g - \int_{D_+^\alpha} div(f\nabla\phi_\lambda)\, dv_g. \tag{2.6}$$

To obtain a lower bound of α-level sets of Steklov eigenfunctions, we need to choose some appropriate test functions. Inspired by the idea in [SZ], it turns out that $f \equiv 1$ and $f = \sqrt{1+|\nabla\phi_\lambda|^2}$ are good choices. Let $f \equiv 1$. We are able to establish the following proposition.

Proposition 2.3.1. *There exists a positive constant $K(\mathcal{N})$ such that, for $\lambda > K(\mathcal{N})$,*

$$\int_{L_\lambda^\alpha} |\nabla\phi_\lambda|\, ds \geq \frac{\lambda^2}{4}\|\phi_\lambda\|_{L^1(\mathcal{M})}.$$

Proof. Since $f = 1$ in (2.6), we have

$$2\int_{L_\lambda^\alpha} |\nabla\phi_\lambda|\, ds = \int_{D_-^\alpha} \triangle\phi_\lambda\, dv_g - \int_{D_+^\alpha} \triangle\phi_\lambda\, dv_g. \tag{2.7}$$

From Lemma 2.2.1, we know that

$$\sqrt{-\triangle_g} = \Lambda + P_0,$$

where $P_0 \in OPS^0(\mathcal{M})$. It follows that

$$-\triangle = \Lambda^2 + P_1 + P_0^2,$$

where $P_1 = \Lambda P_0 + P_0 \Lambda \in OPS^1(\mathcal{M})$. Therefore,

$$\begin{aligned} \triangle\phi_\lambda &= -(\Lambda^2 + P_1 + P_0^2)\phi_\lambda \\ &= -\lambda^2\phi_\lambda - P_1\phi_\lambda - P_0^2\phi_\lambda. \end{aligned}$$

Substituting the above identity into (2.7) implies that

$$\begin{aligned} 2\int_{L_\lambda^\alpha} |\nabla\phi_\lambda|\, ds &= -\int_{D_-^\alpha} \lambda^2\phi_\lambda - \int_{D_-^\alpha} P_1\phi_\lambda - \int_{D_-^\alpha} P_0^2\phi_\lambda \\ &\quad + \int_{D_+^\alpha} \lambda^2\phi_\lambda + \int_{D_+^\alpha} P_1\phi_\lambda + \int_{D_+^\alpha} P_0^2\phi_\lambda \\ &\geq -\lambda^2\int_{D_-^\alpha}(\phi_\lambda - \alpha) + \lambda^2\int_{D_+^\alpha}(\phi_\lambda - \alpha) + \alpha\lambda^2(vol(D_+^\alpha) - vol(D_-^\alpha)) \\ &\quad - \int_{\mathcal{M}} |P_1\phi_\lambda| - \int_{\mathcal{M}} |P_0^2\phi_\lambda| \\ &= \lambda^2\|\phi_\lambda - \alpha\|_{L^1(\mathcal{M})} + \alpha\lambda^2(vol(D_+^\alpha) - vol(D_-^\alpha)) \\ &\quad -(1+\lambda)\|\tilde{P}_0\phi_\lambda\|_{L^1(\mathcal{M})} - \|P_0^2\phi_\lambda\|_{L^1(\mathcal{M})}, \end{aligned} \tag{2.8}$$

where $\tilde{P}_0 = P_1(1+\Lambda)^{-1} \in OPS^0(\mathcal{M})$. Now there are three "bad" terms in (2.8):

$$\alpha\lambda^2(vol(D_+^\alpha) - vol(D_-^\alpha)), \quad \|\tilde{P}_0\phi_\lambda\|_{L^1(\mathcal{M})}, \quad \|P_0^2\phi_\lambda\|_{L^1(\mathcal{M})}.$$

We are going to estimate each of them.

For the term $\alpha\lambda^2(vol(D_+^\alpha) - vol(D_-^\alpha))$, we can not get a better way, but assume that $|\alpha| \le \epsilon(\mathcal{N})\lambda^{-\frac{n-1}{4}}$ for some small $\epsilon(\mathcal{N})$ depending only on $\mathcal{N}$. We will determine it later on.

For the other two "bad" terms, we are able to control them by the L^1 norm of ϕ_λ multiplied by an ϵ power of λ. We can establish the following lemma.

Lemma 2.3.2. *Let $P \in OPS^0(\mathcal{M})$. Then for any positive constant ϵ, there exists $C = C(\mathcal{N}, \epsilon)$ such that*

$$\|P\phi_\lambda\|_{L^1(\mathcal{M})} \le C\lambda^\epsilon \|\phi_\lambda\|_{L^1(\mathcal{M})}.$$

Proof. Let $\delta > 0$. By Hölder's inequality,

$$\|P\phi_\lambda\|_{L^1(\mathcal{M})} \lesssim \|P\phi_\lambda\|_{L^{1+\delta}(\mathcal{M})} \lesssim \|\phi_\lambda\|_{L^{1+\delta}(\mathcal{M})},$$

where we have used Lemma 2.2.3. As we know,

$$\begin{aligned} \|\phi_\lambda\|_{L^{1+\delta}(\mathcal{M})} &\le \|\phi_\lambda\|_{L^\infty(\mathcal{M})}^{\frac{\delta}{1+\delta}} \|\phi_\lambda\|_{L^1(\mathcal{M})}^{\frac{1}{1+\delta}} \\ &= \Big(\frac{\|\phi_\lambda\|_{L^\infty(\mathcal{M})}}{\|\phi_\lambda\|_{L^1(\mathcal{M})}}\Big)^{\frac{\delta}{1+\delta}} \|\phi_\lambda\|_{L^1(\mathcal{M})}. \end{aligned} \tag{2.9}$$

Thanks to Lemma 2.2.2, we get

$$\|\phi_\lambda\|_{L^\infty(\mathcal{M})} \lesssim \lambda^{\frac{n-1}{2}}.$$

By (2.2),

$$\|\phi_\lambda\|_{L^1(\mathcal{M})} \gtrsim \lambda^{-\frac{n-1}{4}}.$$

Thus, from (2.9), we have

$$\|\phi_\lambda\|_{L^{1+\delta}(\mathcal{M})} \lesssim \lambda^{\frac{3(n-1)\delta}{4(1+\delta)}} \|\phi_\lambda\|_{L^1(\mathcal{M})}.$$

Selecting δ so small that $\frac{3(n-1)\delta}{4(1+\delta)} \leq \epsilon$, we are done. □

With aid of Lemma 2.3.2, we continue the proof of Proposition 2.3.1. Let's go back to (2.8). Choosing $\epsilon = 1/2$ in Lemma 2.3.2, we obtain

$$\begin{aligned} 2\int_{L_\lambda^\alpha} |\nabla\phi_\lambda|\, ds &\geq \lambda^2\|\phi_\lambda - \alpha\|_{L^1(\mathcal{M})} - \alpha\lambda^2 vol(\mathcal{M}) - C(1+\lambda)^{\frac{3}{2}}\|\phi_\lambda\|_{L^1(\mathcal{M})} \\ &\geq \lambda^2\|\phi_\lambda\|_{L^1(\mathcal{M})} - 2\alpha\lambda^2 vol(\mathcal{M}) - C(1+\lambda)^{\frac{3}{2}}\|\phi_\lambda\|_{L^1(\mathcal{M})}. \end{aligned}$$

Since we have assumed that $|\alpha| \leq \epsilon(\mathcal{N})\lambda^{-\frac{n-1}{4}}$, then

$$\begin{aligned} 2\int_{L_\lambda^\alpha} |\nabla\phi_\lambda|\, ds &\geq \lambda^2\|\phi_\lambda\|_{L^1(\mathcal{M})} - 2\epsilon(\mathcal{N})\lambda^{-\frac{n-1}{4}+2} vol(\mathcal{M}) \\ &\quad -C(1+\lambda)^{\frac{3}{2}}\|\phi_\lambda\|_{L^1(\mathcal{M})}. \end{aligned} \tag{2.10}$$

From (2.2), we can choose $\epsilon(\mathcal{N})$ so small that

$$2\epsilon(\mathcal{N})\lambda^{-\frac{n-1}{4}+2} vol(\mathcal{M}) \leq \frac{\lambda^2}{4}\|\phi_\lambda\|_{L^1(\mathcal{M})}.$$

By (2.10) and choosing λ appropriately large which depends only on $\mathcal{N}$, we finally arrive at

$$2\int_{L_\lambda^\alpha} |\nabla\phi_\lambda|\, ds \geq \frac{\lambda^2}{2}\|\phi_\lambda\|_{L^1(\mathcal{M})}.$$

We are done with the proof of Proposition 2.3.1. □

Next we select the test function $f = \sqrt{1 + |\nabla\phi_\lambda|^2}$. We are able to prove the following proposition.

Proposition 2.3.3. *There exists positive constant $C = C(\mathcal{N})$ such that*

$$\int_{L_\lambda^\alpha} |\nabla\phi_\lambda|^2\, ds \leq C(1+\lambda)^3. \tag{2.11}$$

Proof. Let $f = \sqrt{1+|\nabla\phi_\lambda|^2}$ in (2.6). We derive that

$$\begin{aligned}
2\int_{L_\lambda^\alpha} \sqrt{1+|\nabla\phi_\lambda|^2}|\nabla\phi_\lambda|\, ds &= \int_{D_-^\alpha} div(\sqrt{1+|\nabla\phi_\lambda|^2}\nabla\phi_\lambda)\, dv_g \\
&\quad -\int_{D_+^\alpha} div(\sqrt{1+|\nabla\phi_\lambda|^2}\nabla\phi_\lambda)\, dv_g \\
&\leq \int_{\mathcal{M}} |div(\sqrt{1+|\nabla\phi_\lambda|^2}\nabla\phi_\lambda)|\, dv_g \\
&\lesssim \int_{\mathcal{M}} (1+|\nabla\phi_\lambda|^2)^{-1/2}|\nabla^2\phi_\lambda||\nabla\phi_\lambda|^2\, dv_g \\
&\quad +\int_{\mathcal{M}} (1+|\nabla\phi_\lambda|^2)^{1/2}|\triangle\phi_\lambda|\, dv_g.
\end{aligned}$$

Furthermore, we get

$$\begin{aligned}
\int_{L_\lambda^\alpha} |\nabla\phi_\lambda|^2\, ds &\lesssim \int_{\mathcal{M}} (1+|\nabla\phi_\lambda|^2)^{1/2}|\nabla^2\phi_\lambda|\, dv_g \\
&\lesssim \Big(\int_{\mathcal{M}} (1+|\nabla\phi_\lambda|^2\, dv_g\Big)^{\frac{1}{2}}\Big(\int_{\mathcal{M}} |\nabla^2\phi_\lambda|^2\, dv_g\Big)^{\frac{1}{2}} \\
&\lesssim (1+\lambda)^3, \qquad (2.12)
\end{aligned}$$

where Lemma 2.2.3 has been used in last inequality. $\square$

We are ready to give the proof of Theorem 2.1.1. We use an idea in [HS] by Hezari and Sogge.

Proof of Theorem 2.1.1. On one hand, by Proposition 2.3.3,

$$\begin{aligned}\int_{L_\lambda^\alpha} |\nabla \phi_\lambda|\, ds &\leq (\int_{L_\lambda^\alpha} |\nabla \phi_\lambda|^2\, ds)^{\frac{1}{2}} |L_\lambda^\alpha|^{\frac{1}{2}} \\ &\lesssim \lambda^{\frac{3}{2}} |L_\lambda^\alpha|^{\frac{1}{2}}.\end{aligned} \tag{2.13}$$

On the other hand, from Proposition 2.3.1, we have

$$\begin{aligned}\int_{L_\lambda^\alpha} |\nabla \phi_\lambda|\, ds &\geq \frac{\lambda^2}{4} \|\phi_\lambda\|_{L^1(\mathcal{M})} \\ &\gtrsim \lambda^{2-\frac{n-1}{4}},\end{aligned} \tag{2.14}$$

where we have used Lemma 2.2.2 in the last inequality. Combining the estimates (2.13) and (2.14), we arrive at

$$|L_\lambda^\alpha| \gtrsim \lambda^{\frac{3-n}{2}}$$

with $\lambda \geq K(\mathcal{N})$ and $|\alpha| \leq \epsilon(\mathcal{N})\lambda^{-\frac{n-1}{4}}$. $\square$

Chapter 3

Restrictions of eigenfunctions to submanifolds

3.1 Introduction

In this paper, we will concentrate on restrictions of eigenfunctions to totally geodesic submanifolds. Before we present our theorem, let's review the estimates in [BGT] about eigenfucntions restricted to general submanifolds:

Theorem 3.1.1. *(N. Burq, P. Gérard, and N. Tzvetkov [BGT])*

Let (M, g) be a compact smooth Riemannian manifold of dimension d, and let Σ be a smooth submanifold of dimension k. There exists a constant $C > 0$ such that for any φ_λ, we have

$$\|\varphi_\lambda\|_{L^p(\Sigma)} \leq C(1+\lambda)^{\rho(k,d)}\|\varphi_\lambda\|_{L^2(M)}$$

where

$$\rho(d-1,d) = \begin{cases} \frac{d-1}{2} - \frac{d-1}{p} & \text{if } p_0 = \frac{2d}{d-1} < p \leq +\infty \\ \frac{d-1}{4} - \frac{d-2}{2p} & \text{if } 2 \leq p < p_0 = \frac{2d}{d-1} \end{cases} \tag{3.1}$$

$$\rho(d-2,d) = \frac{d-1}{2} - \frac{d-2}{p} \quad \textit{if } 2 < p \leq +\infty \tag{3.2}$$

$$\rho(k,d) = \frac{d-1}{2} - \frac{k}{p} \quad \textit{if } 1 \leq p \leq d-3 \tag{3.3}$$

If $p = p_0 = \frac{2d}{d-1}$ and $k = d-1$, we have

$$\|\varphi_\lambda\|_{L^p(\Sigma)} \leq C(1+\lambda)^{\frac{d-1}{2d}} \log^{\frac{1}{2}}(\lambda) \|\varphi_\lambda\|_{L^2(M)}$$

and if p=2 and k=d-2, we have

$$\|\varphi_\lambda\|_{L^p(\Sigma)} \leq C(1+\lambda)^{\frac{1}{2}} \log^{\frac{1}{2}}(\lambda) \|\varphi_\lambda\|_{L^2(M)}$$

Remark 3.1.2. These estimates are sharp in the general case except $(d,k,p) = (d, d-1, \frac{2d}{d-1})$ and $(d,k,p) = (d, d-2, 2)$, which has a log loss. Later on, Rui Hu gave another proof in [RH] of these estimates and showed the log loss for the case $(d,k,p) = (d, d-1, \frac{2d}{d-1})$ can be removed. For the remaining case, in [ChenS], they showed that if $d = 3$ and the submanifold is a geodesic, then the log loss can be removed. Here we deal with the general case of $(d,k,p) = (d, d-2, 2)$, where $d > 3$ and the submanifold is totally geodesic.

The following is our main result:

Theorem 3.1.3. *(Main Theorem)*

Let (M,g) be a compact smooth Riemannian manifold of dimension d, and let Σ be a smooth totally geodesic submanifold of dimension $d-2$. There exists a constant $C > 0$ such that for any φ_λ, we have

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \leq C(1+\lambda)^{\frac{1}{2}} \|\varphi_\lambda\|_{L^2(M)}$$

3.2 Preliminaries

In this section, we will review and prepare some general results needed in the proof of the main theorem. First, we need the Hadamard parametrix, see [CSF] or [CSHZ] for references.

Lemma 3.2.1. *(Hadamard parametrix) Let (M, g) be a compact without boundary. If $t \le \rho$, here $\rho > 0$ and smaller than the injective radius of (M, g). If $N > n + 3$, then we have:*

$$(\cos t\sqrt{-\triangle_g})(x; y) = K_N(t, x; y) + R_N(t, x; y) \tag{3.4}$$

where $R_N \in C^{N-n-3}([-\rho, \rho] \times M \times M)$, and

$$K_N(t, x; y) = \begin{cases} \partial_t(\sum_{\nu=1}^{N} \omega_\nu(x, y)E_\nu(t, \kappa(x, y))) & \text{if } t \ge 0 \\ -\partial_t(\sum_{\nu=1}^{N} \omega_\nu(x, y)E_\nu(-t, \kappa(x, y))) & \text{if } t < 0 \end{cases} \tag{3.5}$$

Here $\kappa(x, y)$ is the vector from x to y in the local geodesic coordinates at x. And $\omega_\nu \in C^\infty(M \times M)$, specifically $\omega_0(x, x) = 1, \forall x \in M$.

E_ν are distributions such that

$$\partial_t E_0(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \cos(t|\xi|)d\xi, \tag{3.6}$$

and $E_\nu, \nu = 1, 2, 3...$ is a finite linear combination of Fourier integrals of the form:

$$H(t)t^j(2\pi)^{-n} \int_{\mathbb{R}^n \setminus B_1(0)} e^{ix\cdot\xi \pm it|\xi|}|\xi|^{-\nu-1-k}d\xi + \eta_{j\nu} \tag{3.7}$$

where $j + k = \nu$ and $\eta_{j\nu}$ are smooth.

Remark 3.2.2. We will also use the property that $\omega_0(x, x) = 1, \forall x \in M$ in our proof.

We also need the following lemma in [BGT](Prop 6.3), which will be used several times in the proof.

Lemma 3.2.3. *(N. Burq, P. Gérard, and N. Tzvetkov [BGT]) Let (N, h) be a compact Riemannian manifold, $dimN = k$. Q_λ is an operator with kernel*

$$Q_\lambda(x, y) = \sum_{\pm} \frac{e^{\pm i\lambda d_h(x,y)}}{(\lambda d_h(x, y))^m} a^\pm(x, y, \lambda), \lambda d_h(x, y) > 1 \tag{3.8}$$

and $|Q_\lambda(x, y)| \leq C$. Here $a^\pm(x, y, \lambda) \in C^\infty(M \times M \times \mathbb{R})$, $\partial^\alpha_{x,y} a^\pm \leq C_\alpha$. Then

$$\begin{aligned} ||Q_\lambda||_{L^2(N)\to L^2(N)} &\lesssim \lambda^{-m-\frac{k-1}{2}} \sum_{j\leq \log\lambda} 2^{j(m-\frac{k+1}{2})} \\ &\lesssim \begin{cases} \lambda^{-k} & \text{if } m > \frac{k+1}{2} \\ \lambda^{-m-\frac{k-1}{2}} & \text{if } m < \frac{k+1}{2} \\ \lambda^{-m-\frac{k-1}{2}} \log\lambda & \text{if } m = \frac{k+1}{2} \end{cases} \end{aligned} \tag{3.9}$$

3.3 Proof

Without loss of generality, we assume the injective radius of (M, g) is greater than 10. Choose any $\chi \in \mathcal{S}(\mathbb{R})$, such that $\chi(0) = 1$, $Supp\hat{\chi} \subset [1, 2]$. Let $\chi_\lambda f = \chi(\lambda - \sqrt{-\triangle_g})f$, then $\chi_\lambda\varphi_\lambda = \varphi_\lambda$. Thus it suffices to show

$$||\chi_\lambda||_{L^2(M)\to L^2(\Sigma)} \lesssim \lambda^{\frac{1}{2}} \tag{3.10}$$

By TT^* argument, 3.10 is equivalent to

$$||\chi_\lambda\chi^*_\lambda||_{L^2(\Sigma)\to L^2(\Sigma)} \lesssim \lambda \tag{3.11}$$

Denote $T_\lambda = \chi_\lambda \chi_\lambda^*$. A simple calculation shows the kernel of $\chi_\lambda \chi_\lambda^*$ is the same as

$$\chi^2(\lambda - \sqrt{-\triangle_g})(x,y)|_{\Sigma\times\Sigma} \tag{3.12}$$

Let $\phi = \chi^2$, then $\phi(0) = 1, Supp\hat{\chi} \subset [2,4]$.

$$\begin{aligned} T_\lambda &= \phi(\lambda - \sqrt{-\triangle_g}) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\phi}(t) e^{i(\lambda - \sqrt{-\triangle_g})t} \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \hat{\phi}(t) e^{-i\lambda t}\cos(t\sqrt{-\triangle_g}) - \phi(\lambda + \sqrt{-\triangle_g}) \end{aligned} \tag{3.13}$$

By Hadamard parametrix, as in Lemma 3.2.1, the kernel of T_λ is

$$\begin{aligned} T_\lambda(x,y) &= \frac{\omega_0(x,y)}{\pi}\int_{\mathbb{R}} \hat{\phi}(t)e^{i\lambda t}\cdot(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\kappa(x,y)\cdot\xi}\cos(t|\xi|)d\xi dt \\ &\quad + \int_{\mathbb{R}} \hat{\phi}(t)e^{i\lambda t}\sum_{\nu=1}^{N}\sum_{j=1}^{\nu} a_{vj}^{\pm}\omega_\nu(x,y)\partial_t(H(t)t^j(2\pi)^{-n}\int_{\mathbb{R}^n\setminus B_1(0)} e^{ix\cdot\xi\pm it|\xi|}|\xi|^{-2\nu-1+j}d\xi) \\ &\quad + R_N(x,y,\lambda) - \phi(\sqrt{-\triangle_g}+\lambda)(x,y) \end{aligned} \tag{3.14}$$

$$\begin{aligned} &= \omega_0(x,y)\int_{\mathbb{R}^n}\phi(\lambda - |\xi|)e^{i\kappa(x,y)\cdot\xi}d\xi \\ &\quad + \sum_{\nu=1}^{N}\sum_{j=1}^{\nu}\omega_\nu(x,y)\int_{\mathbb{R}^n\setminus B_1(0)}\phi_{1j\nu}^{\pm}(\lambda\pm|\xi|)e^{i\kappa(x,y)\cdot\xi}|\xi|^{-2\nu-1+j}d\xi \\ &\quad + \sum_{\nu=1}^{N}\sum_{j=1}^{\nu}\omega_\nu(x,y)\int_{\mathbb{R}^n\setminus B_1(0)}\phi_{2j\nu}^{\pm}(\lambda\pm|\xi|)e^{i\kappa(x,y)\cdot\xi}|\xi|^{-2\nu+j}d\xi \\ &\quad + \omega_0(x,y)\int_{\mathbb{R}^n}\phi(\lambda+|\xi|)e^{i\kappa(x,y)\cdot\xi}d\xi + \tilde{R}_N(x,y,\lambda) - \phi(\sqrt{-\triangle_g}+\lambda)(x,y) \end{aligned} \tag{3.15}$$

Here $\phi_{1j\nu}^{\pm}$ is the inverse Fourier transform of $(2\pi)^{-n+1}a_{j\nu}^{\pm}j\cdot t^{j-1}\hat{\phi}(t)$, and $\phi_{2j\nu}^{\pm}$ is the inverse Fourier transform of $(2\pi)^{-n+1}a_{j\nu}^{\pm}t^j\hat{\phi}(t)\cdot(\pm i)$, which are also Schwartz functions independent with λ.

Next we introduce a new operator which will play a key role in the proof.

Define S_r^ν, $\nu = 0, 1, 2, 3...$ to be the operator with kernel:

$$S_r^\nu(x, y) = \omega_\nu(x, y) \int_{S^{n-1}(1)} e^{ir\kappa(x,y)\cdot\omega} d\omega \tag{3.16}$$

By Stationary phase, see [LH1] or [CSF], we can see that S_r^ν satiesfies the condition in Lemma 3.2.3 with $k = n - 2$ and $m = \frac{n-1}{2}$. Thus by Lemma 3.2.3, we have the following estimate:

$$||S_r^\nu||_{L^2(\Sigma)\to L^2(\Sigma)} \lesssim \lambda^{-n+2} \log \lambda \tag{3.17}$$

Now let's go back to 3.15, there are 5 terms in it and the last three terms can be easily controlled. For the first two terms, by using spherical coordinates for the ξ variables, we can rewrite them as

$$\begin{aligned}
\Phi_\lambda(x, y) &= \int_0^\infty \phi(\lambda - r) S_r^0(x, y) r^{n-1} dr \\
&\quad + \sum_{\nu=1}^{N} \sum_{j=1}^{\nu} \int_1^\infty \phi_{1j\nu}^\pm(\lambda \pm r) S_r^\nu(x, y) r^{-2\nu-1+j} \cdot r^{n-1} dr \\
&\quad + \sum_{\nu=1}^{N} \sum_{j=1}^{\nu} \int_1^\infty \phi_{2j\nu}^\pm(\lambda \pm r) S_r^\nu(x, y) r^{-2\nu+j} \cdot r^{n-1} dr \\
&= A_\lambda + B_\lambda
\end{aligned}$$

By using the above estimate 3.17, we are able control the second term as follows:

$$
\begin{aligned}
||B_\lambda||_{L^2(\Sigma)\to L^2(\Sigma)} &\lesssim \sum_{\nu=1}^{N}\sum_{j=1}^{\nu}\int_1^\infty (|\phi^\pm_{1j\nu}(\lambda\pm r)| + |\phi^\pm_{2j\nu}(\lambda\pm r)|)||S_r^\nu||_{L^2(\Sigma)\to L^2(\Sigma)} r^{-2\nu+j}\cdot r^{n-1}dr \\
&\lesssim \sum_{\nu=1}^{N}\sum_{j=1}^{\nu}\int_1^\infty (|\phi^\pm_{1j\nu}(\lambda\pm r)| + |\phi^\pm_{2j\nu}(\lambda\pm r)|)||S_r^\nu||_{L^2(\Sigma)\to L^2(\Sigma)} r^{-1}\cdot r^{n-1}dr \\
&\lesssim \sum_{\nu=1}^{N}\sum_{j=1}^{\nu}\int_1^\infty |\phi^\pm_{j\nu}(\lambda\pm r)| r^{-n+2}\log r\cdot r^{n-2}dr \\
&\lesssim \log\lambda \qquad (3.18)
\end{aligned}
$$

The same procedure will be used several times.

Similarly, if we can show the following stronger estimate without the log loss for S_r^0, then we are able to control A_λ as needed, and the proof will be done.

Lemma 3.3.1. *we have*

$$
||S_r^0||_{L^2(\Sigma)\to L^2(\Sigma)} \lesssim r^{-n+2}. \tag{3.19}
$$

Proof. Since this estimate is a local estimate, without loss of generality, we can assume that Σ is closed. Let $h = g|_\Sigma$, then (Σ, h) is a closed Riemannian manifold.

Let $\tilde{\kappa}(x,y) : \Sigma\times\Sigma \to \mathbb{R}^{n-2}$, be the vector from x to y in the local geodesic coordinates with respect to (Σ, h) at x. Since Σ is totally geodesic, we can assume $\kappa|_{\Sigma\times\Sigma} = (\tilde{\kappa}, 0, 0)$. Accordingly, we can make the following change of coordinates:

$$
B^{n-2}(1)\times[0,2\pi) \to S^{n-1} : (z, \sqrt{1-|z|^2}\cos\theta, \sqrt{1-|z|^2}\sin\theta) \tag{3.20}
$$

The Jacobian is 1, thus we can modify the kernel of operator S_r as

$$\begin{aligned} S_r(x,y) &= \omega_0(x,y)\int_{B^{n-2}(1)}\int_0^{2\pi} e^{ir\kappa(x,y)\cdot\omega(z,\theta)}d\theta dz \\ &= 2\pi\omega_0(x,y)\int_{B^{n-2}(1)} e^{ir\tilde{\kappa}(x,y)\cdot z}dz \\ &= 2\pi\omega(x,y)r^{-n+2}\int_{B^{n-2}(r)} e^{i\tilde{\kappa}(x,y)\cdot z}dz \end{aligned} \tag{3.21}$$

Let $\overline{S}_r$ be the operator with kernel

$$\overline{S}_r(x,y) = \omega_0\int_{B^{n-2}(r)} e^{i\tilde{\kappa}(x,y)\cdot z}dz \tag{3.22}$$

Then 3.19 is equivalent to

$$||\overline{S}_r||_{L^2(\Sigma)\to L^2(\Sigma)} \lesssim 1 \tag{3.23}$$

To prove this estimate, we compare $\overline{S}_r$ to an operator with uniform bound over $L^2(\Sigma)$. Consider the eigenfunctions and eigenvalues of $-\triangle_h$ over Σ:

$$-\triangle_h e_{\mu_j} = \mu e_{\mu_j}, 0 = \mu_0 < \mu_1 \le \mu_2 \le ... \tag{3.24}$$

Define P_μ to be the projection map to the eigenspace with eigenvalue $\le \mu$, that is

$$P_\mu = \sum_{\mu_j\le\mu} E_{\mu_j} = \chi_{[-\mu,\mu]}(\sqrt{-\triangle_h}) \tag{3.25}$$

Obviously, $||P_\mu||_{L^2(\Sigma)\to L^2(\Sigma)} \leq 1$. The kernel of P_μ is given by

$$
\begin{aligned}
P_\mu(x,y) &= (2\pi)^{-1}\int_{\mathbb{R}} e^{it\sqrt{-\triangle_h}}\hat{\chi}_{[-\mu,\mu]}(t)dt \\
&= (\pi)^{-1}\int_{\mathbb{R}} e^{it\sqrt{-\triangle_h}}\frac{\sin\mu t}{t} \\
&= (\pi)^{-1}\int_{\mathbb{R}} e^{it\sqrt{-\triangle_h}}\beta(t)\frac{\sin\mu t}{t}dt + (\pi)^{-1}\int_{\mathbb{R}} e^{it\sqrt{-\triangle_h}}(1-\beta(t))\frac{\sin\mu t}{t}dt \qquad (3.26)
\end{aligned}
$$

Here $\beta(t)$ is an even cut off function supported in $[-\delta,\delta]$, and $\delta>0$ is less than the injective radius of (Σ,h). Let r_μ be the inverse Fourier transform of $(1-\beta(t))\frac{2\sin\mu t}{t}$, as in [CSF], r_μ satisfies

$$
|r_\mu(t)| \leq C_N(1+||t|-\mu|)^{-N}, \mu \geq 1, N=1,2,3... \qquad (3.27)
$$

Hence we can rewrite 3.26 as

$$
P_\mu(x,y) = (\pi)^{-1}\int_{\mathbb{R}} \beta(t)\frac{\sin\mu t}{t}\cos t\sqrt{-\triangle_h}dt + r_\mu(\sqrt{-\triangle_h}) \qquad (3.28)
$$

The second term is a multiplier uniformly bounded over $L^2(\Sigma)$. For the first term, we can compute it by the Hadamard parametrix about (Σ, h):

$$
\begin{aligned}
P_\mu(x,y) &= \frac{\tilde{\omega}_0(x,y)}{\pi}\int_{\mathbb{R}} \cdot (2\pi)^{-n}\int_{\mathbb{R}^{n-2}} \beta(t)\frac{\sin \mu t}{t} e^{i\tilde{\kappa}(x,y)\cdot z}\cos t|z| dtdz \\
&\quad + \int_{\mathbb{R}} \beta(t)\frac{\sin \mu t}{t} e^{i\mu t}\partial_t(\sum_{\nu=1}^{N}\tilde{\omega}_\nu(x,y)E_\nu(t,\tilde{\kappa}(x,y))) \\
&\quad + \tilde{R}_N(x,y,\mu) \\
&= \frac{\tilde{\omega}_0(x,y)}{\pi}\int_{\mathbb{R}} \cdot (2\pi)^{-n}\int_{\mathbb{R}^{n-2}} \beta(t)\frac{\sin \mu t}{t} e^{i\tilde{\kappa}(x,y)\cdot z} e^{it|z|} dtdz \\
&\quad + \int_{\mathbb{R}} \beta(t)\frac{\sin \mu t}{t} e^{i\mu t}\partial_t(\sum_{\nu=1}^{N}\tilde{\omega}_\nu(x,y)E_\nu(t,\tilde{\kappa}(x,y))) \\
&\quad + \tilde{R}_N(x,y,\mu) \\
&= \tilde{\omega}_0(x,y)(2\pi)^{-n}\int_{\mathbb{R}^{n-2}} \chi_{[-\mu,\mu]} e^{i\tilde{\kappa}(x,y)\cdot z} dz \\
&\quad + \tilde{\omega}_0(x,y)(2\pi)^{-n}\int_{\mathbb{R}^{n-2}} r_\mu(|z|) e^{i\tilde{\kappa}(x,y)\cdot z} dz \\
&\quad + \int_{\mathbb{R}} \beta(t)\frac{\sin \mu t}{t} e^{i\mu t}\partial_t(\sum_{\nu=1}^{N}\tilde{\omega}_\nu(x,y)E_\nu(t,\tilde{\kappa}(x,y))) \\
&\quad + \tilde{R}_N(x,y,\mu) \\
&= \tilde{A}_\mu + \tilde{B}_\mu + \tilde{C}_\mu + \tilde{D}_\mu
\end{aligned}
$$

We can see that the kernel of $\tilde{A}_\mu$ is very close to $\overline{S}_r$. If we can show $\tilde{A}_\mu$ is uniformly bounded, then $\overline{P}_\mu = (2\pi)^n\tilde{A}_\mu$ is also uniformly bounded. let $\mu = r$ and consider the difference between $\overline{S}_r$ and $\overline{P}_r$:

$$
\overline{S}_r(x,y) - \overline{P}_r(x,y) = (\omega_0(x,y) - \tilde{\omega}_0(x,y))\int_{B^{n-2}(r)} e^{i\tilde{\kappa}(x,y)\cdot z} dz \tag{3.29}
$$

Since $\omega_0(x,x) = \tilde{\omega}_0(x,x) = 1$, we have $\omega_0(x,y) - \tilde{\omega}_0(x,y) = O(d_h(x,y))$, thus by stationary phase and Lemma 3.2.3, we know

$$
||\overline{S}_r - \overline{P}_r||_{L^2(\Sigma)\to L^2(\Sigma)} \lesssim 1 \tag{3.30}
$$

this implies

$$||\overline{S}_r||_{L^2(\Sigma)\to L^2(\Sigma)} \lesssim 1 \tag{3.31}$$

Now we show $\tilde{A}_\mu$ is uniformly bounded. From

$$P_\mu = \tilde{A}_\mu + \tilde{B}_\mu + \tilde{C}_\mu + \tilde{D}_\mu \tag{3.32}$$

we know we only need to show $\tilde{B}_\mu$, $\tilde{C}_\mu$ and $\tilde{D}_\mu$ are uniformly bounded.

For $\tilde{B}_\mu$, using spherical coordinates, we know that

$$\tilde{B}_\mu \;=\; (2\pi)^{-n}\int_0^\infty r_\mu(\rho)\rho^{n-3}\tilde{\omega}_0(x,y)\int_{\mathbb{S}^{n-3}(1)} e^{i\tilde{\kappa}(x,y)\cdot\tilde{\omega}}d\tilde{\omega}d\rho \tag{3.33}$$

by Stationary Phase and Lemma 3.2.3, we have

$$||B_\mu||_{L^2(\Sigma)\to L^2(\Sigma)} \lesssim \int_0^\infty (1+|\mu-\rho|)^{-N}\rho^{n-3}\cdot\rho^{-\frac{n-3}{2}-\frac{n-2-1}{2}}d\rho \lesssim 1. \tag{3.34}$$

For $\tilde{C}_\mu$, notice

$$\sin\mu t = \frac{1}{2}(e^{i\mu t} - e^{-i\mu t}) \tag{3.35}$$

similarly to [CSF](Chapter 5), we can rewrite the kernel of $\tilde{C}_\mu$ as

$$\begin{aligned}\tilde{C}_\mu(x,y) \;&=\; \sum_{\nu=1}^{N}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-2}} e^{i\tilde{\kappa}(x,y)\cdot\eta\pm it(|\eta|-\mu)}a_{1\nu}^{\pm}(t,x,|\eta|)d\eta dt\\ &\; \sum_{\nu=1}^{N}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-2}} e^{i\tilde{\kappa}(x,y)\cdot\eta\pm it(|\eta|+\mu)}a_{2\nu}^{\pm}(t,x,|\eta|)d\eta dt\end{aligned} \tag{3.36}$$

where $a_1\nu^\pm$ and $a_1\nu^\pm$ are symbols of order $-\nu$, by Stationary Phase we can see that its kernel satisfies the following condition:

$$\tilde{C}_\mu(x,y) = \sum_{\nu=1}^{N} \mu^{n-2-\nu} \sum_{\pm} \frac{e^{\pm i\mu d_h(x,y)}}{(\mu d_h(x,y))^{\frac{n-3}{2}}} a_\nu^\pm(x,y,\mu) + O(\mu^{-N}), \mu d_h(x,y) > 1 \tag{3.37}$$

and $|\tilde{C}_\mu(x,y)| \leq C\mu^{n-3}$. Here $a_\nu^\pm(x,y,\mu) \in C^\infty(M \times M \times \mathbb{R})$, $\partial_{x,y}^\alpha a_\nu^\pm \leq C_{\nu\alpha}$. Hence, by Lemma 3.2.3, we have $\tilde{C}_\mu$ is also uniformly bounded on $L^2(\Sigma)$.

The uniform boundedness of $\tilde{D}_\mu$ over $L^2(\Sigma)$ is obvious. And this completes the proof.

$\square$

Bibliography

[BGT] N. Burq, P. Ge'rard, and N. Tzvetkov, Restrictions of the Laplace-Beltrami Eigenfunctions to Submanifolds, Duke Mathematical Jounal, Vol. 138, No. 3, 2007

[BL] K. Bellova and F.-H. Lin, Nodal sets of Steklov eigenfunctions, arXiv:1402.4323.

[ChenS] Xuehua Chen, Christopher D. Sogge, A Few Endpoint Geodesic Restriction Estimates for Eigenfunctions, Communications in Mathematical Physics, July 2014, Volume 329, Issue 2, pp 435-459

[CM] T. H. Colding and W. P. Minicozzi II: Lower bounds for nodal sets of eigenfunctions, Comm. Math. Phys. 306(2011), 777-784.

[CS1] C. D. SOGGE, Concerning the Lp norm of spectral clusters for second-order elliptic operators on compact manifolds, J. Funct. Anal. 77 (1988), 123-138. MR 0930395

[CS3] Sogge, C.D. Kakeya-Nikodym averages and Lp-norms of eigenfunctions. Tohoku Math. J. 63 (2011),519-538.

[CSF] C.D. Sogge, Fourier integrals in classical analysis, Cambridge Tracts in Mathematics, 105. Cambridge University Press, Cambridge, 1993.

[CSHZ] C. D. SOGGE, Hangzhou Lectures on Eigenfunctions of the Laplacian (AM-188), Annals of Mathematics Studies, 2014.

[DF] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds, Invent. Math. 93(1988), 161-183.

[DF2] H. Donnelly, C. Fefferman, Nodal sets of eigenfunctions on surfaces, J. Amer. Math. Soc.,3(1990), 333-353.

[Do] Dong, R.T. (1992). Nodal sets of eigenfunctions on Riemann surfaces. J. Diff. Geom. 36:493-506.

[HL] Q. Han and F. H. Lin: Nodal sets of solutions of Elliptic Differential Equations, book in preparation (online at http://www.nd.edu/qhan/nodal.pdf).

[HS] H. Hamid, C.D. Sogge, a natural lower bound for the size of nodal sets, Anal. PDE. 5(2012), no. 5, 1133-1137.

[HS1] R. Hardt, L. Simon, Nodal sets for solutions of elliptic equations, J. Differential Geometry, 30(1989), 505-522.

[JB] J. Brüning, Über Knoten von Eigenfunctionen des Laplace Beltrami Operator, Math. Z, 158(1978), 15-21.

[J.B] J. Bourgain, Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces, Israel Journal of Mathematics, January 2013, Volume 193, Issue 1, pp 441-458

[LH1] Lars Hörmander, The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis

[LH4] Lars Hörmander, The Analysis of Linear Partial Differential Operators IV: Fourier Integral Operators

[M] D. Mangoubi, A remark on recent lower bounds for nodal sets, Comm. Partial Differential Equations 36(2011), no. 12, 2208-2212.

[RH] Rui Hu, L^p norm estimates of eigenfunctions restricted to submanifolds, Forum Math. 21 (2009), 1021-1052 DOI 10.1515/FORUM.2009.051

[SS] A. Seeger and C.D. Sogge, Bounds for eigenfunctions of differential operators, Indiana Math. J. 38(1989), 669-682.

[SZ] C.D. Sogge and S. Zelditch, Lower bounds on the Hausdorff measure of nodal sets, Math. Res. Lett. 18(2011), 25-37.

[SZ1] C.D. Sogge and S. Zelditch, Lower bounds on the Hausdorff measure of nodal sets II, Math. Res. Lett. 19(2012), no.6, 1361-1364.

[SZ2] Sogge, C. D. ; Zelditch, S., Riemannian manifolds with maximal eigenfunction growth, Duke Math. J. 114 (2002), 387-437.

[SZ3] C. D. Sogge and S. Zelditch, Focal points and sup-norms of eigenfunctions, arXiv:1311.3999.

[SZ4] C.D. Sogge and S. Zelditch, Focal points and sup-norms of eigenfunctions on analytic Riemannian manifoles II: the two-dimensional case, arXiv:1409.2063

[T] M. Taylor, Partial differential equations II. Qualitative studies of linear equation, Applied mathematicsal Sciences 116, Springer-Verlag, New York, 1996.

[WZ] Xing Wang and Jiuyi Zhu, A Lower bound for the nodal sets of Steklov eigenfunctions, arxiv 1411.0708.

[Zel] S. Zelditch, Measure of nodal sets of analytic steklov eigenfunctions, arXiv:1403.0647

Vitae

Xing Wang was born on November 13, 1986 in Longhui city, Hunan province, China. He received his Bachelor of Science in Mathematics and Applied Mathematics from Zhejiang University in Jun 2010. He was accepted into a Doctoral Program at The Johns Hopkins University in the fall of 2010. He received his Master of Arts in Mathematics from The Johns Hopkins University in May 2011. His dissertation was completed under the guidance of Dr. Christopher D. Sogge and this dissertation was defended on March 6, 2015.